

## Mathematics 100A Quiz 5

### Due: Friday 4 December 2020

**Instructions:** This is a closed note, closed book quiz. Please write clearly and fully explain your solutions. You have 30 minutes from when you begin to read the questions to complete it.

1. Suppose that a finite group  $G$  acts on a set  $S$  with two orbits,  $\mathcal{O}_1$  and  $\mathcal{O}_2$ . Assume that  $|\mathcal{O}_1| = 5$  and  $|\mathcal{O}_2| = 4$ . Is it possible that  $G$  has order 30? Be sure to prove your answer.

*Proof.* One has  $|\mathcal{O}_1|$  divides  $|G|$  and  $|\mathcal{O}_2|$  divides  $|G|$ . So,  $|G|$  must be divisible by 20. But 30 is not divisible by 20 so  $|G| = 30$  is impossible.  $\square$

2. Denote by  $C_n$  the cyclic group of order  $n$  and  $C_2 = \{1, y\}$  the cyclic group of order 2. Let  $G = (C_n \times C_n) \rtimes C_2$ , where  $y \in C_2$  acts on  $C_n \times C_n$  by exchanging the coordinates, i.e.,  $y * (a, b) = (b, a)$  for  $a, b \in C_n$ . Prove that if  $g \in G$ , then  $g^{2n} = 1$ .

*Proof.* Every element  $g \in G$  is of the form  $((a, b), 1)$  or  $((a, b), y)$ . If  $g$  is of the first form, then  $g^n = ((a^n, b^n), 1) = ((1, 1), 1) = 1$ , so  $g^{2n} = 1$  as well. If  $g$  is of the second form, then

$$g^2 = ((a, b), y)((a, b), y) = ((a, b)y * (a, b), y^2) = ((ab, ba), 1).$$

Then  $g^{2n} = (g^2)^n = ((ab, ba), 1)^n = ((1, 1), 1) = 1$  as desired.  $\square$